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The Matrix of a Finite Rotation in the Space of Trees

G. I. KUZNETSOV

Kurchatov Institute of Atomic Energy, Moscow

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Abstract

The matrix of a finite rotation (i.e., the matrix of a "symmetric top" in many-dimensional space) has been constructed, describing the transformation of higher spherical harmonics under rotation of coordinates.

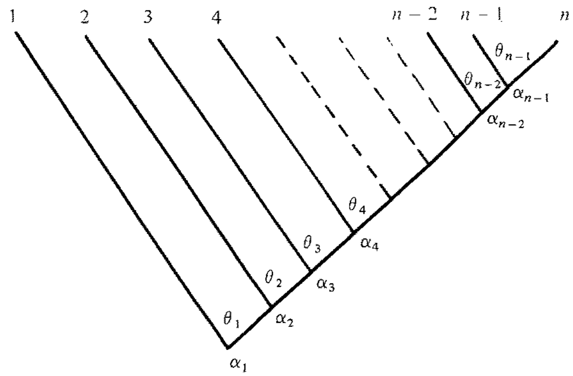
As is known, the matrix of a rotation of three-dimensional space induces the Wigner D-function that carries out the transformation of spherical harmonics defined on the two-dimensional sphere. The aim of the present work is to construct a function that is generated by the rotation of n-dimensional space and carries out the transformation of the so-called "tree function," i.e., the higher spherical harmonics defined on the $(n-1)$-dimensional sphere (Vilenkin et al., 1965).

In the rotation group of n-dimensional space one can choose $n(n-1)/2$ different one-parametric subgroups, for example, those that correspond to the transformation of coordinates X_i and X_j $(X_i \neq X_j)$ only and leave unchanged all the other coordinates (Gel'fand and Tseitlin, 1950). The matrices of such one-parametric transformations can be brought to the form

$$g_{ik}(t) = \begin{pmatrix}
1 & 0 & \cdots & 0 & \cdots & \overset{(i)}{0} & \cdots & \overset{(k)}{0} & \cdots & & 0 \\
0 & 1 & \cdots & & \cdots & & \cdots & & \cdots & & 0 \\
\vdots & & & & & & & & & & \vdots \\
0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & & \cdots & & 0 \\
\vdots & & & & & & & & & & \vdots \\
0 & \cdots & & & \cdots & \cos t & \cdots & \sin t & 0 & \cdots & 0 \\
\vdots & & & & & & & & & & \vdots \\
\vdots & & & & & & & & & & \vdots \\
0 & \cdots & & & \cdots & -\sin t & \cdots & \cos t & 0 & \cdots & 0 \\
\vdots & & & & & & & & & & \vdots \\
0 & \cdots & & & & & & \cdots & 0 & 1 & 0 \\
0 & \cdots & & & & & & \cdots & 0 & 0 & 1
\end{pmatrix}
\begin{matrix} \\ \\ \\ \\ \\ (i) \\ \\ \\ (k) \\ \\ \\ \\ \end{matrix}
\qquad (1)$$

For the sake of convenience we shall construct the matrix of finite rotations in the space of trees that correspond to the canonical reduction of the space (Vilenkin et al., 1965). This matrix enables us to construct matrices of rotation in any space of trees because the transition to other types of trees is known (Kil'dyushov, 1972; Kil'dyushov and Kuznetsov, 1973).

According to Vilenkin et al. (1965) one can bring the canonical reduction of the space $R_n \supset R_{n-1} \supset \cdots \supset R_1$ in correspondence with the tree given below and the following solution of Laplace equation:



$$\stackrel{\text{def}}{=} \prod_{i=1}^{n-2} \{N_{n_i}^{l_{i+1} l_{i+1}}\}^{-1/2} (1 - y_i^2)^{\alpha_{i+1}/2} P_{n_i}^{l_{i+1} l_{i+1}}(y_i) \times \frac{\exp(i\alpha_{n-1}\theta_{n-1})}{(2\pi)^{1/2}} = \prod_{i=1}^{n-2} \psi_i(y_i) \frac{\exp(i\alpha_{n-1}\theta_{n-1})}{\sqrt{2\pi}} \tag{2}$$

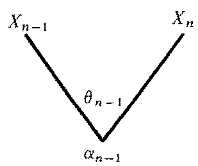
Here $P_\kappa^{\alpha\beta}(x)$ is the Jacobi polynomial (when $\alpha = \beta$ it degenerates into the Gegenbauer polynomial), $N_\kappa^{\alpha\beta}$ is a square of its norm

$$N_\kappa^{\alpha\beta} = \frac{2^{\alpha+\beta+1}\Gamma(\kappa+\alpha+1)\Gamma(\kappa+\beta+1)}{(2\kappa+\alpha+\beta+1)\Gamma(\kappa+1)\Gamma(\kappa+\alpha+\beta+1)}$$

α_i are separation constants of the Laplace equation,

$$n_i = \alpha_i - \alpha_{i+1}; \qquad y_i = \cos\theta_i; \qquad l_i = 2j_c + 1 = \alpha_i + (n - i - 1)/2$$

The normed solution $\exp(i\alpha_{n-1}\theta_{n-1})/(2\pi)^{1/2}$ corresponds to the "fork" formed with X_{n-1} and X_n coordinates, i.e.,



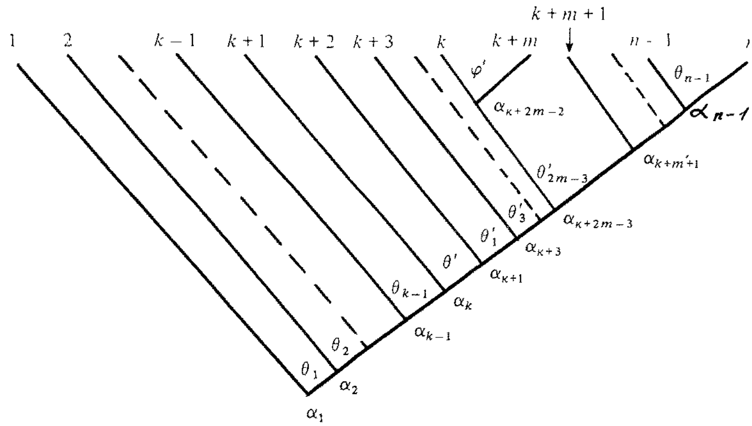
$$\stackrel{\text{def}}{=} \exp(i\alpha_{n-1}\theta_{n-1})/(2\pi)^{1/2} \tag{3}$$

It is clear that under rotation in $X_{n-1}X_n$ plane the "fork" (3) will be multiplied by an exponential factor. Hence in order to construct the transformation matrix of the tree function that arises under rotation of the coordinates it is necessary to construct a "fork" using the coordinates in the plane in which the rotation takes place (this operation is equivalent to a transplantation of branches (Kil'dyushov, 1972; Kil'dyushov and Kuznetsov, 1973) and a transition to another tree), to carry out the rotation at angle φ and then, using the reverse transplantation, to return to the original tree.

Let us consider rotation at angle φ in $X_k X_{k'}(k' = k + m)$ plane. Carrying out in sequence the transformation of the kth branch from its original place up to the $(k+m)$th place (see Kil'dyushov, 1972; Kil'dyushov and Kuznetsov, 1973; Kuznetsov and Smorodinsky, 1975a) we obtain the following formula:

$$\psi_{\text{can}}^{l_1 \cdots l_{n-1}}(y_1 \cdots y_{n-1}) = \sum_{\substack{l_{\kappa+1}, l_{\kappa+3} \cdots l_{\kappa+2m-3}, \\ l_{\kappa+2m-2}}} \times \begin{pmatrix} l_k l_{k+1} \cdots l_{k+m+1} & \\ & l_{\kappa+2m-2} \\ l_{\kappa+1} l_{\kappa+3} \cdots l_{\kappa+2m-3} & \end{pmatrix} \psi_{\text{per}\; l_{\kappa+2m-2}\,;\, l_{k+m+1} \cdots l_{n-1}}^{l_1 \cdots l_k;\, l_{\kappa+1} \cdots l_{\kappa+2m-3}} \times (y_1 \cdots y_{k-1}; y', y'_{\kappa+1} \cdots y'_{\kappa+2m-3}, \varphi', y_{k+m+1} \cdots y_{n-1}) \tag{4}$$

where $\psi_{\text{can}}^{l_1 \cdots l_{n-1}}(y_1 \cdots y_{n-1})$ is defined by (2) and $\psi_{\text{per}\,\{l\}}^{\{l\}}(\{y\})$ is given by the relation



$$\stackrel{\text{def}}{=} \prod_{i=1}^{\kappa-1} \psi_i(y_i) \{N_{\alpha_k - \alpha_{\kappa+1}}^{l_{\kappa+1} l_{\kappa+1}}\}^{-1/2} [1 - (y^1)^2]^{\alpha_{\kappa+1}/2} P_{\alpha_k - \alpha_{\kappa+1}}^{l_{\kappa+1} l_{\kappa+1}}(y')$$

$$\times \prod_{j=3,5\ldots}^{2m-3} \{N_{\alpha_{\kappa+j-2} - \alpha_{\kappa+j}}^{l_{\kappa+j} l_{\kappa+j}}\}^{-1/2} [1 - (y'_{j-2})^2]^{\alpha_{\kappa+j}/2} P_{\alpha_{\kappa+j-2} - \alpha_{\kappa+j}}^{l_{\kappa+j} l_{\kappa+j}}(y'_{j-2})$$

$$\times\, 2^{(n-k-m+2)/4}\{N_{n'}^{l_s l_c}\}^{-1/2}(1-y_{\kappa+2m-3})^{(\alpha_{k+m+1})/2}$$

$$\times\,(1+y_{\kappa+2m-3})^{|\alpha_{\kappa+2m-2}|/2}P_{n'}^{l_s l_c}(y_{\kappa+2m-3})$$

$$\times\,\frac{\exp(i\alpha_{\kappa+2m-2}\varphi')}{(2\pi)^{1/2}}\prod_{i=k+m+1}^{n-2}\psi_i(y_i)\,\frac{\exp(i\alpha_{n-1}\theta_{n-1})}{(2\pi)^{1/2}} \tag{5}$$

where

$$y'=\cos\theta',\qquad l_{\kappa+i}=\alpha_{\kappa+i}+(n-k-i-1)/2,\qquad y_i'=\cos\theta_i',$$

$$y_{\kappa+2m-3}=\cos 2\theta_{\kappa+2m-3},\qquad l_s=\alpha_{k+m+1}+(n-k-m-2)/2$$

$$l_c=|\alpha_{\kappa+2m-2}|,\qquad n'=(\alpha_{\kappa+2m-3}-\alpha_{\kappa+2m-2}-\alpha_{\kappa+m+1})/2$$

Comparing (2) with (5), we see that the functions corresponding to these trees differ only at the portion from the kth to the $(k+m)$th branch for the tree (2) and from the $(k+1)$th to the $(k+m)$th branch for the tree (5). This means that in order to calculate the transition matrix corresponding to (4) it is sufficient to calculate the integral of the overlapping between these portions of trees.

Using the results given in Kil'dyushov (1972), Kil'dyushov and Kuznetsov (1973), and Kuznetsov and Smorodinsky (1975a) the transition matrix corresponding to (4) can be brought to the following form (j-representation):

$$\begin{pmatrix} j_k j_{k+1}\cdots j_{k+m+1} & \\ j_{\kappa+1}j_{\kappa+3}\cdots j_{\kappa+2m-3} & j_{\kappa+2m-2}\end{pmatrix}=\underbrace{\sum_{j_\kappa j_{\kappa+2}\cdots j_{\kappa+2m-4}}}_{m-1-\text{parameters}}$$

$$\times\left\|\begin{matrix}-\frac{3}{4} & -\frac{3}{4} & j_{k+m+1}\\ j_{\kappa+2m-2} & j_{\kappa+2m-3} & j_{k+m}\end{matrix}\right\|$$

$$\times\prod_{i=2}^{m}\left\|\begin{matrix}-\frac{3}{4} & -\frac{3}{4} & j_{k+i}\\ j_{\kappa+2i-4} & j_{\kappa+2i-5} & j_{k+i-1}\end{matrix}\right\|(-1)^{(2j_\kappa+2i-4+1)/2}$$

$$\times\left\|\begin{matrix}-\frac{3}{4} & -\frac{3}{4} & j_{k+i}\\ j_{\kappa+2i-4} & j_{\kappa+2i-5} & j_{\kappa+2i-3}\end{matrix}\right\| \tag{6}$$

where

$$\left\|\begin{matrix} j_1 & j_2 & j_3\\ j_{12} & j & j_{23}\end{matrix}\right\|$$

are the so called T coefficients (Kil'dyushov, 1972; Kil'dyushov and

Kuznetsov, 1973) which in our case are up to the phase the Clebsch-Gordan coefficients, i.e.,

$$\left\| \begin{matrix} -\frac{3}{4} & -\frac{3}{4} & j_3 \\ j_{12} & j & j_{23} \end{matrix} \right\| = (-)^{j-3j_{23}+j_{12}-2j_3-5/4} C^{2j_{23}+1/2,\,2j_3+1}_{jj_3-j_{12};\,jj_3+j_{12}+1}$$

In general the T coefficients turn out to be $6j$ symbols analytically continued to nonphysical (from the point of view of the momentum theory) domain of j's (Kuznetsov and Smorodinsky, 1975b). According to definition the momentum $j_{\kappa-1}$ is j_k.

Applying the operator $R_{k\,k+m}(\varphi)$ [operator of the rotation at angle φ in the k, $(k+m)$ plane] to the function $\psi^{l_1\cdots l_{n-1}}_{\text{can}}(y_1\cdots y_{n-1})$ and taking into the account its relation to the function

$$\psi^{l_1\cdots l_k,\,l_{\kappa+1}\cdots l_{\kappa+2m-3}}_{\text{per}\,l_{\kappa+2m-2},\,l_{k+m+1}\cdots l_{n-1}}(y_1\cdots y_{k-1};\{y'\},\varphi')$$

[see (4)] we find the following expression for the rotation matrix:

$$R^{\{j_{k+1}\cdots j_{k+m}\},\,\{j'_{k+1}\cdots j_{k+m}\}}_{k,\,k+m}(\varphi) = \sum_{\substack{j_{\kappa+1},j_{\kappa+3}\cdots j_{\kappa+2m-3},\\ j_{\kappa+2m-2}}}$$

$$\times \begin{pmatrix} j_k \cdots j_{k+m+1} & & \\ & & j_{\kappa+2m-2} \\ j_{\kappa+1} j_{\kappa+3} \cdots j_{\kappa+2m-3} & & \end{pmatrix}$$

$$\times \exp(il_{\kappa+2m-2}\varphi) \begin{pmatrix} j_k, & j'_{k+1} \cdots j'_{k+m}, j_{k+m+1} & \\ & & j_{\kappa+2m-2} \\ j_{\kappa+1} & j_{\kappa+3} \cdots j_{\kappa+2m-3} & \end{pmatrix}^{*} \qquad (7)$$

The symbols $(\because\because\cdot)$ in (7) are defined in (6), i.e., are one of the types of $3nj$ symbols.

The product of $n(n-1)/2$ such matrices gives the function of "symmetric top" in the space of $(2\alpha_1+n-2)(n+\alpha_1-3)!/(n-2)!\,\alpha_1!$ dimensions, i.e., matrix of finite rotations in the space of trees.

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